# Multiplicative stochastic resonance

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A multiplicative bistable stochastic system perturbed by a periodic forcing term is shown to exhibit stochastic resonance with increasing intensity of the multiplicative noise. Such an effect is related to the phenomenon of stochastic stabilization, which takes place in the unperturbed system.

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The amplitude of the periodic component of the output signal from a periodically modulated bistable system in the presence of *additive* external noise, increases with the noise intensity up to a maximum value when the rate of the noise-induced switch process approaches the forcing frequency. Such a phenomenon, termed *stochastic resonance* (SR) [1], was investigated by a number of authors [1–8] and was applied in many areas of natural sciences [9].

There exist, however, many cases of physical interest [10] where the role of fluctuating control parameter is played by a *multiplicative* noise [10–13]. In the present paper, we show that SR may be obtained by tuning the multiplicative noise intensity, as well. The enhancement of the periodic component of the output signals is traced back to the phenomenon of *stochastic stabilization,* which would occur in the absence of periodic modulation [10,11].

A stationary variation of the class of stochastic processes addressed here is the noise-assisted escape over fluctuating barriers [14–16]. Recently, Doering and Gadoua [15] discovered that under quite general conditions such an escape mechanism exhibits a peculiar resonant behavior: the relevant escape rate approaches a maximum with increasing correlation time of the barrier fluctuations. A more general treatment of this phenomenon is discussed in Ref. [16]. The complicated interplay of additive and multiplicative (random) perturbations are at the basis of the observed resonant behaviors of both the stationary activation processes of Refs. [15,16] and the nonstationary switch processes we report on.

We investigated the phenomenon of SR in the overdamped bistable system described by the following stochastic differential equation

$$\dot{x} = -V'(x) + x\xi_M(t) + \xi_A(t) + A\cos\omega t \ , \tag{1}$$

with

$$V(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4 \tag{2}$$

by means of analog simulation [17]. The fluctuating parameters $\xi_i(t)$, with $i = A,M$, are stationary zero-mean valued, Gaussian random processes with autocorrelation functions

$$\langle \xi_i(t)\xi_j(0)\rangle = 2Q_i\delta_{ij}\delta(t) \ . \tag{3}$$

The origin of time in Eq. (1) has been fixed arbitrarily. Furthermore, the amplitude of the modulating term is taken small enough for the bistable nature of the process $x(t)$ to be retained, i.e., $|A|x_0 \ll \Delta V$ with $x_0 = \sqrt{a/b}$ and $\Delta V = a^2/4b$. Here, $\pm x_0$ denotes the stable minima and $\Delta V$ the barrier height of the potential (2).

The main conclusions of our simulation work is that the process $x(t)$ is periodically modulated by the external bias $A\cos\omega t$ according to the approximate law

$$\langle x(t)\rangle \simeq \bar{x}_1\cos(\omega t + \phi_1) \ . \tag{4}$$

The amplitude $\bar{x}_1$ and the phase $\phi_1$ of $\langle x(t)\rangle$ depend on $A$, $Q_A$, and $Q_M$, whereas no dc component has been observed for $A \neq 0$, no matter what $Q_A$. Most notably, the amplitude $\bar{x}_1$ shows a typical SR behavior with increasing $Q_M$ (*multiplicative* SR), while keeping $Q_A$ fixed. In Fig. 1 we plot the dependence of $\bar{x}_1$ on $Q_M$ for the most

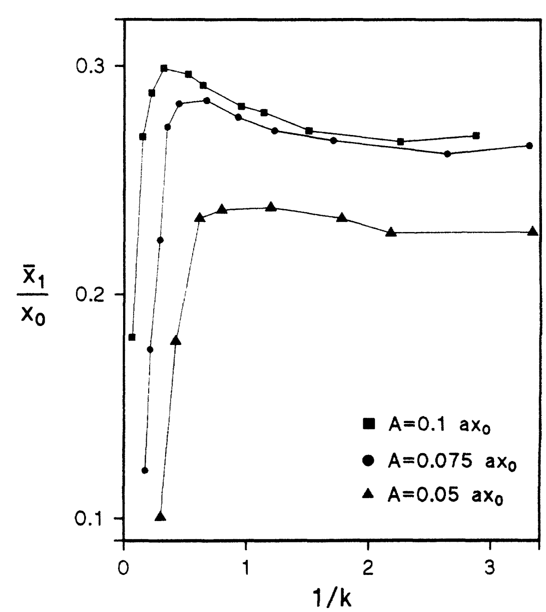


FIG. 1. $\bar{x}_1$ versus $Q_M$ for $Q_A = 0$ and different values of $A$. The potential parameters are $x_0 = 2.2$ V and $a = 10^4$ s$^{-1}$.

 

remarkable case of a purely multiplicative bistable process ($Q_A = 0$).

In order to interpret the outcome of our analog simulations at low forcing frequencies, $\omega \ll a$, it would suffice to determine the properties of the distribution function $P(x, A; t)$, solution of the Fokker-Planck equation

$$\frac{\partial}{\partial t} P = \frac{\partial}{\partial x} \left[ V'(x) - Q_M x - A \cos\omega t + \frac{\partial}{\partial x}(Q_A + Q_M x^2) \right] P \tag{5}$$

associated with Eq. (1).

In the presence of a *static* tilting, i.e., for $\omega = 0$, the stationary solution of Eq. (5) reads

$$P_0(x, A) = N_0(A) \left[ x^2 + \frac{Q_A}{Q_M} \right]^{-1/2 + [1 + (k/2)(Q_A/\Delta V)]} \times \exp\left[ -k\frac{x^2}{x_0^2} - \frac{A}{Q_M |x|} \right] , \tag{6}$$

where $k \equiv a/2Q_M$ and $N_0(A)$ is a suitable normalization constant. Three cases are of particular interest.

(i) $A = Q_A = 0$. The process $x(t)$ is confined on one-half axis at any time, depending on the initial conditions, so that $|\langle x \rangle| > 0$. A sort of noise-induced phase transition [10] occurs for $Q_M = a$. For $Q_M < a$, the relevant distribution function peaks at $x_0(1-2k)^{1/2}$, whereas for $Q_M > a$ it becomes singular at the origin. This effect, termed stochastic stabilization, is characterized by the appearance of long tails in $P_0(x,0)$ [18] for $Q_M \gg a$.

(ii) $A = 0$, $Q_A > 0$. By adding an additive noise, no mater how weak, $x(t)$ diffuses on the entire $x$ axis. The relevant distribution function is then symmetric for $x \to -x$ and $\langle x \rangle \equiv 0$. In the presence of a weak additive noise, $Q_A \ll x_0^2 Q_M$, the distribution function (6) comes fairly close to $P_0(x,0)$ [18] with $N_0(A) = N_0/2$, apart from a small symmetric neighborhood around the origin (where it is always finite [10]).

(iii) $Q_A = 0, A \neq 0$. $P_0(x, A)$ vanishes on the negative half axis for $A > 0$ and vice versa. Accordingly, since $P_0(0, A) \equiv 0$, for $\pm A > 0$ the static distribution function $P_0(x, A)$ shows one peak located at $\pm x_m$, with $x_m(Q_M)$ a decreasing function of $Q_M$. In particular, $x_m(Q_M) \sim x_0 + |A|/2(a - Q_M)$ for $Q_M \ll a$ and $x_m(Q_M) \sim |A|/(Q_M - a)$ for $Q_M \gg a$.

In the presence of a *periodic* tilting, i.e., for $\omega > 0$, the process $x(t)$ is no longer stationary and a time-dependent distribution function $P(x, A; t)$ is required to describe its steady state. However, in the limit of low forcing frequency, assumed throughout our simulation work, the adiabatic approximation $P(x, A; t) \simeq P_0(x, A(t))$ with $A(t) = A \cos\omega t$ suffices to shed light on the nonstationary dynamics underlying the phenomenon of multiplicative SR.

To interpret correctly our experimental results, we recall that, when simulating a purely multiplicative bistable system by means of an analog device, two limitations are unavoidable.

(a) The presence of small additive noise due to ubiquitous fluctuations in the circuitry. As a consequence, our results for $Q_A = 0$ are to be taken with some caution. Unfortunately, a fully analytical treatment of Eq. (5) with both $Q_A > 0$ and $Q_M > 0$ lies beyond our capabilities even for $A = 0$. Numerical algorithms based on continued fraction expansions are the only tool available to date to crack down the problem [12,13]. The escape rate $\mu(Q_A, Q_M)$ between the stable points of $V(x)$ was computed in Ref. [12] for $A = 0$ as a function of the multiplicative noise intensity. For a given value of $Q_A$, $\mu$ grows linearly with increasing $Q_M$ from zero [where $\mu(Q_A, 0)$ coincides with $\mu_k(Q_A)$, the well-known Kramers rate [20]] up to a critical value $Q_M = a/4$ (continuum spectrum threshold [10]) and, then, flattens out until the stochastic stabilization condition $Q_M = a$ is reached. For even larger values of $Q_M, \mu$ diverges faster than exponentially.

(b) Any commercial noise generator is characterized by a finite correlation time (colored noise). The typical autocorrelation function of a simulated noise $\xi_i(t)$ is well approximated by an exponential function, $\langle \xi_i(t)\xi_i(0) \rangle = (Q_i/\tau_i)\exp(-|t|/\tau_i)$, which tends to Eq. (3) for vanishingly small values of $\tau_i$, only. Effects due to the finite correlation time of $\xi_i(t)$ are negligible when $1/\tau_i$ is larger than any other dynamical frequency in the problem under study. For instance, in the unperturbed, purely multiplicative case of Eq. (1) with $Q_A = A = 0$, this amounts to requiring that

$$(a + Q_M)\tau_M \ll 1 . \tag{7}$$

It follows immediately that colored noise effects are bound to show up for large $Q_M/a$ values, i.e., in the regime of stochastic stabilization. Under such circumstances a side peak grows out of the long tail of $P_0(x,0)$ [18] as pointed out in Ref. [13]. Thus, contrary to the ideal white-noise case, the second moment of $P_0(x,0)$ is a slowly increasing function of $Q_M$ with $\langle x^2(Q_M) \rangle \geq x_0^2$.

We are now in the position to explain qualitatively the phenomenon of multiplicative SR observed in the periodically perturbed bistable system (1) with $\omega \ll a$. Let us consider first the purely multiplicative case $Q_A = 0$ (Fig. 1). It is clear from our discussion of Eq. (6), case (iii), that for $Q_A = 0$ the forcing term alone is responsible for $x(t)$ to switch back and forth between the positive and the negative half axis. Should the adiabatic approximation hold good for any value of $Q_M$, the process $x(t)$ would approach instantaneously its most probable value in the vicinity to the peak of the distribution function $P_0(x, A(t))$. Therefore, the amplitude $\bar{x}_1$ of the periodic component of $x(t)$, (4), would be of the order $x_m$, that is a monotonic decreasing function of $Q_M$. However, Fig. 1 shows a dramatic drop of $\bar{x}_1$ as $Q_M$ tends to zero. Such remarkable deviation from the prediction of the adiabatic approximation is due to the fact that with decreasing $Q_M$ the switch time of $x(t)$ between positive and negative values, controlled by $A(t)$ periodically reversing sign, grows much longer than the forcing period. For instance, on assuming that at $t = 0$ $x$ is trapped on the un-

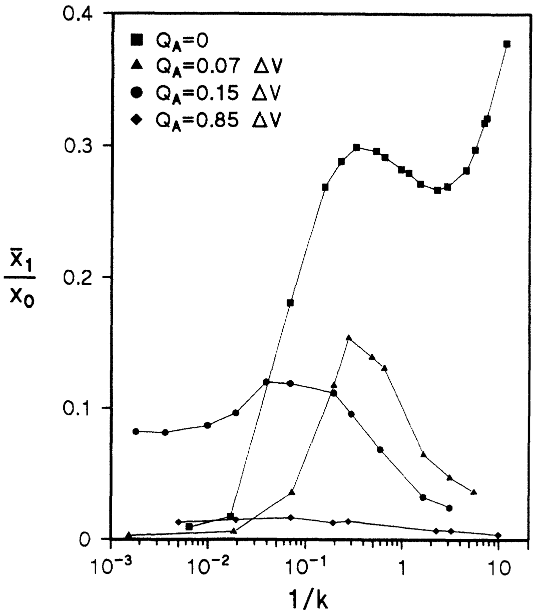


FIG. 2. $\bar{x}_1$ versus $Q_M$ for $A=0.1ax_0$ and different values of $Q_A$. The potential parameters are as in Fig. 1.

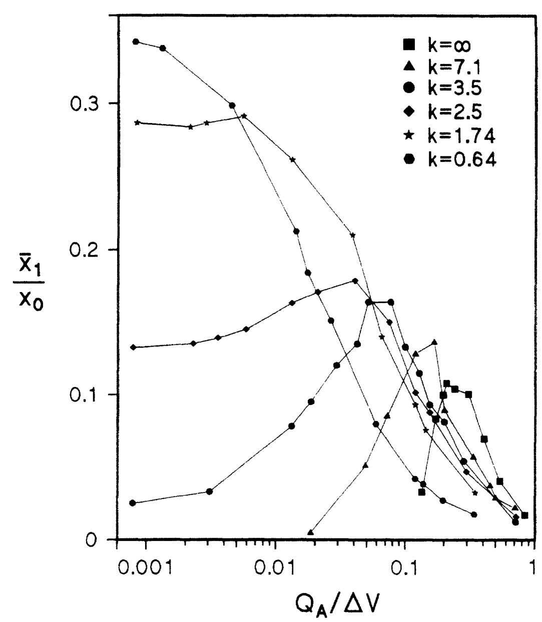


FIG. 3. $\bar{x}_1$ versus $Q_A$ for $A=0.1ax_0$ and different values of $Q_M$. The potential parameters are as in Fig. 1.

stable half axis $x<x_A\simeq A/a$ with $A>0$, the mean-first-passage-time $\tau_A$ required by $x$ to escape through $x_A$ onto the stable half axis $x>x_A$ can be easily calculated [19,20] at the leading order in $(x_0/x_A)^2$, i.e.,

$$\tau_A(Q_M)=\frac{\Gamma(k)}{2ak^k}\left[\frac{x_0}{x_A}\right]^{2k}. \tag{8}$$

The escape time $\tau_A$ is a strongly divergent quantity for $x_A/x_0\to 0$. On increasing $Q_M$ close to the condition of stochastic stabilization, more precisely for $k\leq 1$, such a divergence is substantially weakened, so that the condition for the adiabatic approximation to apply, $\omega\tau_A(Q_M)\ll 1$ may be verified. In this regime the nonstationary process $x(t)$ is mainly controlled by the modulated *interwell* dynamics described by $P_0(x,A(t))$ and, as stated above, $\bar{x}_1$ approaches $x_m$. In the opposite limit, $\omega\tau_A\gg 1$ (i.e., $Q_M\ll a$), the steady-state distribution of $x(t)$ spreads over the entire $x$ axis with oscillating local maxima [19] at $\pm x_0+A(t)/2(a-Q_M)$ (modulated *intrawell* dynamics). It follows immediately that for $Q_M=0+$ the amplitude $\bar{x}_1$ is of the order $A/2a$, that is much smaller than the value of $x_m$ at $k=1$, whence the appearance of the SR peaks of Fig. 1 for $\omega\tau_A\sim 1$. Accordingly, the SR peaks shift to the left with increasing $A$. Finally, it should be noticed that for large values of $Q_M$ the color effects described in (b) become dominant to the extent that the expected decrease of $\bar{x}_1$ for $\tau_M=0$ is no more observable (Fig. 1; see Ref. [19] for further details). In conclusion, the transition from an intrawell to an interwell modulated dynamics is the basic mechanism responsible for multiplicative SR.

The phenomenon of multiplicative SR becomes easily detectable by switching on the additive noise $\xi_A(t)$. In Fig. 2 the dependence of $\bar{x}_1$ on $Q_M$ for several values of $Q_A$ is displayed in order to illustrate the interplay of additive and multiplicative noise. The switch process between the positive and negative half axis in Eq. (1) is now assisted by the additive noise, which makes the relevant switch time $\tau_A$ decrease with increasing $Q_A$ (and always finite). As a consequence, the maxima of the curves $\bar{x}_1(Q_M)$, centered at around the $Q_M$ value that satisfies the condition $\omega\tau_A\simeq 1$, shift to the left with respect to the purely multiplicative case. Furthermore, in the presence of additive noise $x(t)$ diffuses onto the unstable half axis of $P_0(x,A(t))$ even in the adiabatic approximation, i.e., for large $Q_M$ values, provided that $\mu_k(Q_A)\tau_A\gg 1$. This explains why the high-$Q_M$ tails of the curves $\bar{x}_1(Q_M)$ get depressed with increasing $Q_A$. The two competing trends determine rather broad SR peaks as shown in Fig. 2. On further increasing $Q_A$, the SR peaks are washed out completely [19].

Of course, Eq. (1) also exhibits *additive* SR as shown in Fig. 3, where the amplitude $\bar{x}_1$ of the periodic component of $x(t)$ is plotted *versus* $Q_A$ for different values of $Q_M$ [21]. It is well known [1–8] that in the purely additive case $\bar{x}_1$ peaks of noise intensities such that $\mu_k(Q_A)\simeq\omega$. Therefore, one expects that for small $Q_M$ values the resonance condition may be recast as $\mu(Q_A,Q_M)\simeq\omega$, with $\mu$ plotted in Ref. 12. On making use of the fact that $\mu$ grows almost linearly with $Q_M$ in the range $(0,a)$, it is no surprise that, correspondingly, the position of the SR peaks in Fig. 3 appear to shift to lower $Q_A$ values. Finally, on further increasing $Q_M$, the condition is reached that $\bar{x}_1$ becomes a monotonic decreasing function of $Q_A$ [19].

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[1] R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, Tellus **34**, 10 (1982); SIAM J. Appl. Math. **43**, 565 (1983); C. Nicolis, Tellus **34**, 1 (1982).

[2] B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2628 (1988).

[3] L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. Lett. **62**, 349 (1989).

[4] P. Jung and P. Hänggi, Europhys. Lett. **8**, 505 (1989); Phys. Rev. A **41**, 2977 (1990); **44**, 8032 (1991).

[5] C. Presilla, F. Marchesoni, and L. Gammaitoni, Phys. Rev. A **40**, 2105 (1989).

[6] H. Gang, G. Nicolis, and C. Nicolis, Phys. Rev. A **42**, 2030 (1990); Phys. Lett. A **151**, 139 (1990).

[7] L. Gammaitoni, F. Marchesoni, M. Martinelli, L. Pardi, and S. Santucci, Phys. Lett. A **158**, 449 (1991).

[8] T. Zhou, F. Moss, and P. Jung, Phys. Rev. A **42**, 3161 (1990).

[9] For a review, see Proceedings of the NATO Advanced Research Workshop on "Stochastic Resonance," 1992, San Diego (CA) [J. Stat. Phys. **70** (1/2), 1 (1993)].

[10] A. Schenzle and H. Brandt, Phys. Rev. A **20**, 1628 (1979).

[11] R. Graham and A. Schenzle, Phys. Rev. A **25**, 1731 (1982).

[12] S. Faetti, P. Grigolini, and F. Marchesoni, Z. Phys. B **47**, 353 (1982).

[13] P. Jung and H. Risken, Phys. Lett. A **103**, 38 (1984).

[14] J. Maddox, Nature **359**, 771 (1992).

[15] C. R. Doering and J. C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992).

[16] P. Hänngi (unpublished).

[17] F. Marchesoni, E. Menichella-Saetta, M. Pochini, and S. Santucci, Phys. Rev. A **37**, 3058 (1987).

[18] For $A=Q_A=0$, Eq. (6) boils down to the distribution function reported in Refs. [10,11]: $P_0(x,0)=N_0|x|^{-1+2k}\exp(-kx^2/x_0^2)$ with $N_0=2(k/x_0^2)^k/\Gamma(k)$. The relevant distribution moments are $\langle x^n\rangle=(x_0^2/k)^{n/2}\Gamma(k+n/2)/\Gamma(k)$.

[19] L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. Lett. **71**, 3625 (1993).

[20] P. Hänggi, P. Jung, and F. Marchesoni, J. Stat. Phys. **54**, 1367 (1989), and references therein.

[21] A. Bulsara, E. W. Jacobs, T. Zhou, F. Moss, and L. Kiss, J. Theor. Biol. **152**, 531 (1991).